Convergence Proof for Goldberg's Exponential Series

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ABSTRACT

The Goldberg presentation of the formal series for $\log(e^x e^y)$ in noncommutative symbols x and y is shown to converge when x and y are replaced by normed elements in the unit sphere about zero, provided the norm defining the unit sphere is compatible with multiplication.

We begin with a description of the Goldberg exponential formula. Let x and y be noncommuting indeterminates. It is well known that a formal infinite series $z = z(x,y)$ exists such that $e^z = e^x e^y$. Indeed, $z = \log(1+v)$ with $v = e^x e^y - 1$. There are two standard presentations of z in the literature, one due to E. Dynkin (see [1], for example), and one due to K. Goldberg [2]. Our attention in this note will be focused on the Goldberg series. Goldberg obtained recursively generated coefficients for z in the following form.

First, define a sequence of polynomials $G_1(t), G_2(t), \ldots,$ in a single variable t, in this way: $G_1(t) = 1$; then $G_s(t) = s^{-1}(d/dt)\{t(t-1)G_{s-1}(t)\}$ for $s = 2,3,\ldots$. It is easy to verify that $G_s(t)$ has simple roots all lying in the open interval $(-1,1)$ and strictly interlaced by the roots of $G_{s-1}(t)$. Moreover, the roots of $G_s(t)$ are symmetrically situated relative to $\frac{1}{2}$. Now let w denote a word in the letters x, y. Then

$$z(x,y) = x + y + \sum_{n,\, n \geqslant 2} \; \sum_{w,\, |w| = n} g_w w.$$

The inner sum is over all words w with length $|w| = n$, and the outer sum is over all lengths two or more. The symbol g_w denotes Goldberg's coefficient

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on the word w, a rational number. Goldberg's formula is this: If the word w begins with x, say

$$w = w_{x,y} = x^{s_1}y^{s_2}x^{s_3} \cdots (x \vee y)^{s_m}, \qquad s_1, \ldots, s_m \text{ positive},$$

then its coefficient g_w is

$$g_w = \int_0^1 t^{m'}(t-1)^{m''} G_{s_1}(t) G_{s_2}(t) \cdots G_{s_m}(t)\, dt,$$

where $m' = [m/2]$ and $m'' = [(m-1)/2]$, $[\ ]$ denoting the greatest integer of the enclosed number. On the other hand,for the word $w^* = w_{y,x}$ starting with y, Goldberg showed that its coefficient $g_{w^*} = (-1)^{n-1} g_w$. Also, as Goldberg observed, g_w is unchanged if its exponents $s_1, \ldots, s_m$ are permuted, but this fact won't be used.

If w is a word, $w = w_1 w_2 \cdots w_n$, with each w_i an x or a y, let $[w]$ denote the corresponding iterated and left standardized commutator word built from the letters of word w,

$$[w] = [[\cdots[[w_1, w_2], w_3] \cdots], w_n],$$

where as usual the commutator symbol $[w_1, w_2] = w_1 w_2 - w_2 w_1$. For convenience, the comma will often be omitted, and there should be no confusion with the simultaneous use of $[\ ]$ as the greatest integer function.

It is known [1] that the exponent z in $e^z = e^x e^y$ may be written in terms of commutator words, in the following way:

$$z = x + y + \sum_{n \geqslant 2} n^{-1} \sum_{w, |w| = n} g_w [w].$$

We shall call this the commutator version of Goldberg's series (even though it was probably not known to Goldberg).

Goldberg's series is purely formal: if it (in either its noncommutator or its commutator form) is substituted into the Taylor series about 0 for e^z, then manipulated without regard for convergence, the result may be cast into a product of the series for e^x and e^y. We now address the topic of this paper: If we replace x and y by matrices, or more generally by elements from a normed algebra, will the resulting Goldberg series in matrices or algebra elements be a convergent series? The following two part theorem gives conditions under which this holds. In it a norm $\|\cdot\|$ is said to be compatible with associative multiplication if $\|xy\| \leqslant \|x\|\,\|y\|$, and to be compatible with Lie multiplication if $\|[x, y]\| \leqslant \|x\|\,\|y\|$. (Of course, if $\|\cdot\|$ is compatible with associative multiplication, then $2\|\cdot\|$ yields a norm compatible with Lie multiplication.)

THEOREM. *If x and y become normed elements strictly inside the unit sphere about zero, then Goldberg's noncommutator series converges absolutely, provided the unit sphere is defined by a norm compatible with associative multiplication, and his commutator series also converges absolutely, provided the unit sphere is then under a norm compatible with Lie multiplication.*

Two comments are in order. First, it is surprising that this result appears not to be in the literature, at least in the direct form in which we shall establish it, although absolute convergence proofs exist for the Dynkin presentation of z. Second, this result is a stepping stone toward the convergence theorem that was discussed in the author's lecture at the Valencia (1987) matrix conference, which is described at the end of this paper.

Proof. For nonnegative integers a and b, let $I(a, b) = \int_0^1 t^a(1-t)^b\,dt$. Then $I(a, b) = b(a+1)^{-1}I(a+1, b-1)$. Therefore $I(a, b) = a!b!/(a+b+1)!$. Next, we observe (as already was observed in [3]) that $G_s(t)$ is a product of $[(s-1)/2]$ factors $\{t-(\frac{1}{2}-r)\}\{t-(\frac{1}{2}+r)\}$ and perhaps one factor $t-\frac{1}{2}$, where $0<r<\frac{1}{2}$, since the roots of $G_s(t)$ lie in $(0,1)$ and are symmetric relative to $\frac{1}{2}$. For t in $[0,1]$, the quadratic factor here is bounded in modulus by 4^{-1}, and the linear factor by 2^{-1}. From this it follows, for t in $[0,1]$, that $|G_s(t)| \leqslant 2^{-[s-1]}$, since $G_s(t)$ has degree $s-1$. From the integral representation of g_w, we therefore get $|g_w| \leqslant 2^{-(s_1+\cdots+s_m-m)}I(m', m'') = 2^{-(n-m)}I(m', m'')$ where $m' = [m/2]$, $m'' = [(m-1)/2]$, and $n = s_1 + \cdots + s_m$ is the length of w. Thus $|g_w| \leqslant 2^{-(n-m)}\{m \cdot {}_{m-1}C_{m'}\}^{-1}$. Here ${}_KC_L$ denotes the usual binomial coefficient $K!/L!(K-L)!$.

We now count the number of words $w(x, y)$ of length $|w| = n$ which have as exponents variable positive integer values $s_1, \ldots, s_m$ with $s_1 + \cdots + s_m = n$. This is just ${}_{n-1}C_{m-1}$. There are an equal number of terms $w^* = w(y, x)$. The sum of $|g_w/n|$ extended over all words w of length n and which involve m parts therefore is majorized by

$$2 \cdot n^{-1} \cdot {}_{n-1}C_{m-1} \cdot 2^{-(n-m)} \cdot \{m \cdot {}_{m-1}C_{m'}\}^{-1}.$$

Now, the norms we use satisfy $\|xy\| \leqslant \|x\|\,\|y\|$ or $\|[x, y]\| \leqslant \|x\|\,\|y\|$ for normed elements x and y. Therefore if w is a word of length n in letters x

and y with both $\|x\| \leqslant M$, $\|y\| \leqslant M$, for some number M, then $\|w\| \leqslant M^n$, or $\|[w]\| \leqslant M^n$. Consequently, the sum of the norms of the terms in the degree n homogeneous component of Goldberg's commutator series is majorized by

$$2n^{-1}\left\{\sum_{m=1}^{n} {}_{n-1}C_{m-1}\cdot 2^{-(n-m)}\cdot\{m\cdot {}_{m-1}C_{m'}\}^{-1}\right\}\cdot M^n$$

$$= n^{-1}\cdot 2\cdot 2^{-n}\cdot M^n \sum_{m=1}^{n} 2^m\cdot {}_{n-1}C_{m-1}\cdot\{m\cdot {}_{m-1}C_{m'}\}^{-1}$$

$$\leqslant n^{-1}\cdot 2^2\cdot 2^{-n}\cdot M^n \sum_{m=1}^{n} {}_{n-1}C_{m-1} = \frac{2M^n}{n}.$$

Here, in the next to last step, we use the fact that ${}_{m-1}C_{m'}$ is the largest term, or one of the two equal and largest terms, in the binomial expansion of $(1+1)^{m-1}$, an expansion having m terms. Hence $m\cdot {}_{m-1}C_{m'} \geqslant 2^{m-1}$.

From these calculations it follows that the sum of the norms of the individual terms in the Goldberg commutator series is majorized by

$$2\sum_{n=1}^{\infty}\frac{M^n}{n} = -2\log(1-M),$$

and this is finite if $M < 1$, completing the commutator part of the proof. The noncommutator version is similar (just omit the factor n^{-1}) and results in the majorizing series $2\Sigma M^n$, which is finite if $M < 1$. ∎

The planned application of this theorem is to the fact that an infinite series $\rho(x, y)$ of iterated Lie commutators of x and y exists such that $e^x e^y = e^z$ with $z = sxs^{-1} + tyt^{-1}$ and $s = e^{\rho(x,y)}$, $t = e^{\rho(-y,-x)}$. F. Rouviere [4] has proved by analytic Lie theoretic techniques that a series $\rho(x, y)$ exists satisfying these conditions and which converges when x, y are replaced by normed elements near 0. On the other hand, the present author [5] has developed a computational technique that first establishes the existence of a formally correct series $\rho(x, y)$, second constructs it by a computer implementation through to the degree ten terms, and third furnishes strong evidence that a convergence proof exists based solely on combinatorics of words (that is, not using Lie techniques). The present paper is a step toward this latter proof. The details concerning $\rho(x, y)$ will be reported in a later publication.

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